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Difference between Special Relativistic Gravitational Theory and General Relativity: Weak Field Limit

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In the case of weak fields, we compare the gravitational fields and the dynamical equation of a particle deduced from special relativistic gravitational theory with the corresponding results deduced from general relativity. Then both gravitational theories can be tested by experiments.

In the special relativistic gravitational theory of Junhao and Xiang (1990*a,b*), the gravitational force exerted on a particle is a quadratic function of its four-velocity, and then one cannot find a noninertial system in which all particles are in inertial motion, in the general case. In general relativity, over a limited region the effects of gravitation and of being in an accelerated frame of reference are indistinguishable, and the four-acceleration of a particle is also a quadratic function of its four-velocity in curved space-time. But we have no way to affirm whether the space-time is curved or not except by using the trajectory of a particle. Therefore, we cannot judge which theory is correct if experiments prove that the four-gravitational force is a quadratic function of its four-velocity. To answer this question, we need to obtain detailed knowledge of the gravitational force deduced from both theories. Now, the observations of the gravitational waves from the explosion of SN 1987A have been shown to be more advantageous to the special relativistic gravitational theory than to general relativity. But the action of strongly concentrated material is rare, so we need to discuss the weak-field limit, too.

Considering the linear approximation for weak fields, the metric tensor is of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

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where $\eta_{\mu\nu}$ is the constant metric tensor of the special relativistic theory

$$\eta_{\mu\nu}=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{2}$$

and $h_{\mu\nu}$ and their derivatives are small quantities of the first order. In this case, the field equations of general relativity reduce to

$$\tfrac{1}{2}\Box h_{\mu\nu}=\frac{-4\pi G\rho_0}{c^2}\left\{\eta_{\mu\nu}+\frac{2}{c^2}U_\mu^{(G)}U_\nu^{(G)}\right\} \tag{3}$$

where $\Box$ is the special relativistic form of d'Alembert's operator, $U_\nu^{(G)}=\gamma(\mathbf{u}, c)$, $\gamma=(1-u^2/c^2)^{1/2}$, and (G) denotes general relativity.

In special relativistic gravitational theory the gravitational potential tensor satisfies the equation

$$\frac{\partial^2 A_{\mu\nu}}{\partial x_\alpha\,\partial x_\alpha}+\lambda\,\frac{\partial A_{\mu\omega}\,\partial A_{\omega\nu}}{\partial x_\alpha\,\partial x_\alpha}=\frac{-4\pi G\rho_0}{c^2}\left(\frac{2}{c^2}U_\mu^{(S)}U_\nu^{(S)}+\delta_{\mu\nu}\right) \tag{4}$$

where $U_\nu^{(S)}=\gamma(\mathbf{u}, ic)$, and (S) indicates the special relativistic case. Then we have

$$\tfrac{1}{2}h_{\mu\nu}=\begin{pmatrix}A_{ij} & iA_{i4}\\ iA_{4j} & -A_{44}\end{pmatrix} \tag{5}$$

In the linear approximation, both $h_{\mu\nu}$ and $A_{\mu\nu}$ satisfy the same equation, but their physical meanings are different.

Suppose that a homogeneous sphere rotates about the axis x_3 and its mass and radius are M and r_0, respectively. In the case of $\omega r_0 \ll c$, we can describe the rotation of a sphere by the classical method. In the steady rotation case, the potential tensors due to a sphere deduced from the linear approximation of (4) are

$$A_{11}=\frac{GM}{c^2 r}\left\{1+\frac{2\omega^2 r_0^2}{5c^2}\left[1+\frac{r_0^2}{7r^2}\left(-1+\frac{3x_2^2}{r^2}\right)\right]\right\}$$

$$A_{12}=A_{21}=\frac{-6GM\omega^2 r_0^4 x_1 x_2}{35c^4 r^5}$$

$$A_{22}=\frac{GM}{c^2 r}\left\{1+\frac{2\omega^2 r_0^2}{5c^2}\left[1+\frac{r_0^2}{7r^2}\left(-1+\frac{3x_1^2}{r^2}\right)\right]\right\}$$

$$A_{13}=A_{31}=A_{23}=A_{32}=0$$

$$A_{33}=\frac{GM}{c^2 r}$$

$$A_{14}=A_{41}=\frac{-2iGM\omega r_0^2 x_2}{5c^3 r^3}$$

$$A_{24}=A_{42}=\frac{2iGM\omega r_0^2 x_1}{5c^3 r^3}$$

$$A_{34}=A_{43}=0$$

$$A_{44}=\frac{-GM}{c^2 r} \tag{6}$$

where $r>r_0$.

According to the special relativistic theory of gravitation, the four-gravitational force exerted on a particle is

$$K_\nu^{(S)}=m_0 H_{\nu\sigma\rho} U_\sigma^{(S)} U_\rho^{(S)} \tag{7}$$

where m_0 is the mass of the particle and $H_{\nu\sigma\rho}$ must satisfy the relations

$$H_{\nu\sigma\rho}=H_{\nu\rho\sigma} \tag{8}$$

$$H_{\nu\sigma\rho}+H_{\sigma\rho\nu}+H_{\rho\nu\sigma}=0 \tag{9}$$

The relation between the field strength and potential is

$$H_{\nu\sigma\rho}=\frac{\partial A_{\sigma\rho}}{\partial x_\nu}-\frac{1}{2}\frac{\partial A_{\nu\rho}}{\partial x_\sigma}-\frac{1}{2}\frac{\partial A_{\nu\sigma}}{\partial x_\rho} \tag{10}$$

It is important that (10) is an inevitable result of (9).

From general relativity, the dynamical equation of a particle is

$$\frac{dU^{\nu(G)}}{d\tau}=-\Gamma^\nu_{\sigma\rho} U^{\sigma(G)} U^{\rho(G)} \tag{11}$$

In the case of a weak-field approximation, we have

$$-\Gamma^\nu_{\sigma\rho}=\tfrac{1}{2}\eta^{\nu\alpha}(h_{\sigma\rho,\alpha}-h_{\sigma\alpha,\rho}-h_{\rho\alpha,\sigma}) \tag{12}$$

Substituting (5) into (12), we have

$$-\Gamma^l_{\sigma\rho}=\begin{pmatrix} H_{lij}+N_{lij} & i(H_{li4}+N_{li4}) \\ i(H_{l4j}+N_{l4j}) & -(H_{l44}+N_{l44}) \end{pmatrix} \tag{13}$$

$$-\Gamma^4_{\sigma\rho}=-i\begin{pmatrix} H_{4ij}+N_{4ij} & i(H_{4i4}+N_{4i4}) \\ i(H_{44j}+N_{44j}) & -(H_{444}+N_{444}) \end{pmatrix} \tag{14}$$

where

$$N_{\nu\sigma\rho}=\frac{-1}{2}\left(\frac{\partial A_{\nu\rho}}{\partial x_\sigma}+\frac{\partial A_{\nu\sigma}}{\partial x_\rho}\right) \tag{15}$$

The special relativistic theory of gravitation asks that $H_{\nu\sigma\rho}$ satisfies (9); then (10) characterizes this theory. In general relativity, the expression of $\Gamma^{\nu}_{\sigma\rho}$ is invariable. So $N_{\nu\sigma\rho}$ indicates the substantial difference between both gravitational theories.

It has been pointed out that (7) may be expressed in three-dimensional form

$$\mathbf{F}^{(S)}=m\left(\mathbf{E}^{(S)}+\frac{1}{c}\mathbf{u}\times\mathbf{B}^{(S)}+\frac{1}{c}\mathbf{u}\cdot\mathscr{P}^{(S)}+\frac{1}{c^2}\mathbf{u}\times\mathscr{R}\cdot\mathbf{u}\right) \tag{16}$$

In the same way, (11) may be expressed as

$$\mathbf{F}^{(G)}=m\left(\mathbf{E}^{(G)}+\frac{1}{c}\mathbf{u}\times\mathbf{B}^{(G)}+\frac{1}{c}\mathbf{u}\cdot\mathscr{P}^{(G)}+\frac{1}{c^2}\mathbf{u}\times\mathscr{R}\cdot\mathbf{u}\right)+\mathbf{D}^{(G)} \tag{17}$$

where

$$E_i^{(G)}=-c^2(H_{i44}+N_{i44})=E_i^{(S)}+c^2\frac{\partial A_{i4}}{\partial x_4}$$

$$B_i^{(G)}=\frac{i}{2}c^2(H_{jk4}+H_{j4k}-H_{kj4}-H_{k4j}+N_{jk4}+N_{j4k}-N_{kj4}-N_{k4j})$$
$$=\tfrac{4}{3}B_i^{(S)}$$

$$P_{ij}^{(G)}=\frac{i}{2}c^2(H_{ij4}+H_{i4j}+H_{ji4}+H_{j4i}+N_{ij4}+N_{i4j}+N_{ji4}+N_{j4i})$$

$$=-2ic^2\frac{\partial A_{ji}}{\partial x_4} \tag{18}$$

and $x_4=ict$.

If the gravitational field is permanent, then we get

$$E_i^{(G)}=E_i^{(S)}$$
$$B_i^{(G)}=\tfrac{4}{3}B_i^{(S)}$$
$$P_{ij}^{(G)}=0 \tag{19}$$

Consider a homogeneous sphere rotating about the axis x_3. The gravitational field strengths outside the sphere are

$$E_i^{(S)} = \frac{-GM}{r^3}(x_1, x_2, x_3)$$

$$B_i^{(S)} = \frac{-3GM\omega r_0^2}{5cr^5}(3x_1x_3, 3x_2x_3, -x_1^2 - x_2^2 + 2x_3^2)$$

$$P_{ij}^{(S)} = \frac{3GM\omega r_0^2}{5cr^5}\begin{pmatrix} -2x_1x_2 & x_1^2 - x_2^2 & -x_2x_3 \\ x_1^2 - x_2^2 & 2x_1x_2 & x_1x_3 \\ -x_2x_3 & x_1x_3 & 0 \end{pmatrix}$$

$$R_{ij} = \frac{GM}{r^3}\begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$

$$N_{1jl} = \frac{GM}{2c^2r^3}\begin{pmatrix} 2x_1 & x_2 & x_3 \\ x_2 & 0 & 0 \\ x_3 & 0 & 0 \end{pmatrix}$$

$$N_{2jl} = \frac{GM}{2c^2r^3}\begin{pmatrix} 0 & x_1 & 0 \\ x_1 & 2x_2 & x_3 \\ 0 & x_3 & 0 \end{pmatrix}$$

$$N_{3jl} = \frac{GM}{2c^2r^3}\begin{pmatrix} 0 & 0 & x_1 \\ 0 & 0 & x_2 \\ x_1 & x_2 & 2x_3 \end{pmatrix} \quad (20)$$

where we neglect the contribution of the term in proportion to $\omega^2 r_0^2/c^2$. In this case we have

$$\mathbf{D}^{(G)} = \frac{GMmu_r\mathbf{u}}{c^2r^2} \quad (21)$$

CONCLUSION

1. The forces exerted on a rest particle by the permanent gravitational fields deduced from both gravitational theories are the same.

2. In both theories, the force contains terms in u_i and u_iu_j, respectively. In general relativity, the term in u_i of the force must be perpendicular to $\mathbf{u}$. In special relativistic gravitational theory, this term contains a component along the velocity.

3. In special relativistic gravitational theory the term in u_ju_l of the force must be perpendicular to $\mathbf{u}$. In general relativity this term contains a component along the direction of $\mathbf{u}$.

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